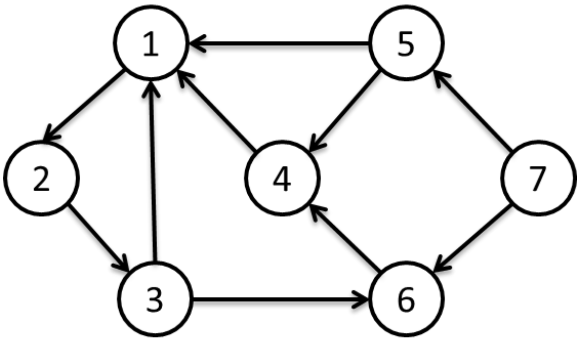
Problem 10

We define the shortest distance from a vertex i to a vertex j on a graph as the number of edges in a path from i to j that contains the smallest number of edges, except that the shortest distance is $+\infty$ when no such path exists and that it is 0 when i and j are identical.

(1) Let us consider the directed graph shown below.



(A) Show the adjacency matrix.

(B) Show a matrix S, whose element $s_{i,j}$ is the shortest distance from a vertex i to a vertex j.

(2) Suppose we are given a simple directed graph $G=(V,E)$, where the vertex set $V=\{1,2,\ldots,n\}$ and E is the edge set. E is represented by a matrix $D^{(0)}=\left(d_{i,j}^{(0)}\right)$, where

$$d_{i,j}^{(0)}=\begin{cases}0 & (\text{if } i=j)\\ 1 & (\text{if an edge } i\to j \text{ exists}).\\ +\infty & (\text{otherwise})\end{cases}$$

(A) Let $V_{i,j}^{(k)}=\{1,2,\ldots,k\}\cup\{i,j\}$. Let $E_{i,j}^{(k)}$ be the set of edges in E that start from and end at a vertex in $V_{i,j}^{(k)}$. Let $d_{i,j}^{(k)}$ be the shortest distance from a vertex i to a vertex j on a directed graph $G_{i,j}^{(k)}=(V_{i,j}^{(k)},E_{i,j}^{(k)})$, and let $D^{(k)}=\left(d_{i,j}^{(k)}\right)$. Express $D^{(1)}$ in terms of $D^{(0)}$.

(B) $D^{(k+1)}$ can be computed from $D^{(k)}$ as follows. Fill in the two blanks.

$$d_{i,j}^{(k+1)}=\min\left(d_{i,j}^{(k)}, \boxed{} + \boxed{}\right)$$

(C) Given G, show an algorithm to compute the all-pair shortest distances, and find its time complexity with regard to n.

(1) (A)

$$
\begin{array}{c|ccccccc|}
 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \\
V_1 & 0 & 1 & +\infty & +\infty & +\infty & +\infty & +\infty \\
V_2 & +\infty & 0 & 1 & +\infty & +\infty & +\infty & +\infty \\
V_3 & 1 & +\infty & 0 & +\infty & +\infty & 1 & +\infty \\
V_4 & 1 & +\infty & +\infty & 0 & +\infty & +\infty & +\infty \\
V_5 & 1 & +\infty & +\infty & 1 & 0 & +\infty & +\infty \\
V_6 & +\infty & +\infty & +\infty & 1 & +\infty & 0 & +\infty \\
V_7 & +\infty & +\infty & +\infty & +\infty & 1 & 1 & 0
\end{array}
$$

(B)

$$
\begin{array}{c|ccccccc|}
 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \\
V_1 & 0 & 1 & 2 & 4 & +\infty & 3 & +\infty \\
V_2 & 2 & 0 & 1 & 3 & +\infty & 2 & +\infty \\
V_3 & 1 & 2 & 0 & 2 & +\infty & 1 & +\infty \\
V_4 & 1 & 2 & 3 & 0 & +\infty & 4 & +\infty \\
V_5 & 1 & 2 & 3 & 1 & 0 & 4 & +\infty \\
V_6 & 2 & 3 & 4 & 1 & +\infty & 0 & +\infty \\
V_7 & 2 & 3 & 4 & 2 & 1 & 1 & 0
\end{array}
$$

(2) (A) $D^{(1)} = \left(d_{i,j}^{(1)}\right) = \min\left(d_{i,j}^{(0)},\ d_{i,1}^{(0)} + d_{1,j}^{(0)}\right)$

$\Rightarrow D^{(1)} = \min\left(D^{(0)},\ E_{i,1} + E_{1,j}\right)$

(B) $d_{i,j}^{(k+1)} = \min\left(d_{i,j}^{(k)},\ d_{i,k+1}^{(k)} + d_{k+1,j}^{(k)}\right)$

(c) floyd (G(V,E)) {

initially $d_{i,j}(0)$ // operation in (2)

for (k=1 ; k<=|V| ; k++)

for(i=1 ; i<=|V| ; i++)

for(j=1 ; j<=|V| ; j++)

$d_{i,j}^{(k)} = \min(d_{i,j}^{(k-1)} , d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)})$

}

time complexity: $O(n^3)$