Use one answer sheet for each of F1–1, F1–2, F2–1, and F2–2.

Q.1 We consider a matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

(1) Derive the inverse matrix of $\mathbf{A}$.
(2) Derive all the eigenvalues of $\mathbf{A}$.
(3) Derive $\mathbf{A}^{10}$.

Q.2 We consider a column vector

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

and a matrix $\mathbf{B}$ that has ten rows and ten columns. $\mathbf{B}(i, j)$ represents the i-th row and the j-th column element of $\mathbf{B}$. All the elements of $\mathbf{B}$ are zero except
$\mathbf{B}(1, 6) = 8$,
$\mathbf{B}(3, 7) = 2$,
$\mathbf{B}(4, 8) = 1/8$,
$\mathbf{B}(6, 4) = 5$,
$\mathbf{B}(7, 1) = 1/4$,
$\mathbf{B}(8, 10) = 4$,
$\mathbf{B}(10, 3) = 1/10$.

Derive $\mathbf{B}^{50}\mathbf{x}$.

(1) $[A \vdots I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(2) Let $Ax = \lambda x$

$$\Rightarrow |A - \lambda E| = \begin{vmatrix} -\lambda & 1 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda(1-\lambda)^2 - 2(1-\lambda) = 0$$

$$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -1$$

(3) ① $\lambda_1 = 1$:

$$|A-E| = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

② $\lambda_2 = 2$

$$|A-2E| = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

③ $\lambda_3 = -1$

$$|A+E| = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow x_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow A[x_1 \ x_2 \ x_3] = [x_1 \ x_2 \ x_3] \, diag(1, 2, -1)$$

Let $[x_1 \ x_2 \ x_3] = P$

$$\Rightarrow A = P \, diag(1, 2, -1) \cdot P^{-1}$$

$$\Rightarrow A^{10} = P (diag)^{10} \cdot P^{-1}$$

$$[P \,\vdots\, E] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{6} & 0 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 2^{10} & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2^{10} & 2 \\ 0 & 2^{11} & -2 \\ 1 & -2^{11} & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2^{10} & 2 \\ 0 & 2^{11} & -2 \\ 1 & -2^{11} & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 \end{bmatrix} = \begin{bmatrix} \frac{2^{10}+2}{3} & \frac{2^{10}-1}{3} & 0 \\ \frac{2^{11}-2}{3} & \frac{2^{11}+1}{3} & 0 \\ \frac{2-2^{11}}{3} & \frac{2-2^{11}}{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 342 & 341 & 0 \\ 682 & 683 & 0 \\ -682 & -682 & 1 \end{bmatrix}$$

$$B=\begin{bmatrix} 0&0&0&0&0&8&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0&\frac{1}{8}&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&5&0&0&0&0&0&0\\ \frac{1}{4}&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&4\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&\frac{1}{10}&0&0&0&0&0&0&0 \end{bmatrix}$$

$$C_{ij} = \sum_{1}^{n} A_{ik} B_{kj}$$

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$$B^7(1,1) = 1$$
$$B^7(3,3) = 1$$
$$B^7(4,4) = 1 \quad \Rightarrow$$
$$B^7(6,6) = 1$$
$$B^7(7,7) = 1$$
$$B^7(8,8) = 1$$
$$B^7(10,10) = 1$$

$$B^{50} \cdot x = (B^7)^7 Bx = Bx$$

$$\Downarrow$$

$$Bx = \begin{bmatrix} 48\\ 0\\ 14\\ 1\\ 0\\ 20\\ \frac{1}{4}\\ 40\\ 0\\ 0.3 \end{bmatrix}$$