Use one answer sheet for each of F1–1, F1–2, F2–1, and F2–2.

Q. Answer the following questions. In the following, n is a positive integer and $\log x$ denotes $\log_e x$.

(1) Let $f(x) = e^x(x^2 + x)$. Derive $\dfrac{d^n f(x)}{dx^n}$.

(2) (i) Find the Maclaurin series of $\log(1 - x)$.

(ii) Derive the following limit:

$$\lim_{x \to 0} \frac{x + \log(1 - x)}{x^2}.$$

(3) The Maclaurin series of e^x is given by $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots$.

Derive the following limit:

$$\lim_{x \to 0} \frac{x \sin x}{e^{x^2} - \cos x}.$$

(1)

$$\frac{df(x)}{dx} = e^x(x^2+x) + e^x(2x+1) = e^x(x^2+3x+1)$$

$$\frac{d^2f(x)}{dx^2} = e^x(x^2+3x+1) + e^x(2x+3) = e^x(x^2+5x+4)$$

$$\frac{d^3f(x)}{dx^3} = e^x(x^2+5x+4) + e^x(2x+5) = e^x(x^2+7x+9)$$

$$\cdots$$

$$\frac{d^nf(x)}{x^n} = e^x\left[x^2+(2n+1)x+n^2\right]$$

(2)(i) $$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)x^2}{2!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$$

$$f'(x) = -(1-x)^{-1}, \quad f^{(2)}(x) = -(1-x)^{-2}$$

$$f^{(3)}(x) = -2(1-x)^{-3} \Rightarrow f^{(n)}(x) = -(n-1)!\cdot(1-x)^{-n}$$

$$\Rightarrow f(x) = 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

$$= -\sum_{k=1}^{n}\frac{x^k}{k}$$

(2) (ii)

$$\lim_{x\to 0} \frac{x+\left[-x-\frac{x^2}{2}-o(x^3)\right]}{x^2} = \lim_{x\to 0} \frac{-\frac{x^2}{2}-o(x^3)}{x^2}$$

$$= -\frac{1}{2}$$

(3) $$\lim_{x\to 0} \frac{x\sin x}{e^{x^2}-\cos x} = \frac{x^2}{1+x^2+o(x^4)-\left(1-\frac{1}{2}x^2+o(x^4)\right)}$$

$$= \lim_{x\to 0} \frac{x^2}{\frac{3}{2}x^2+o(x^4)}$$

$$= \frac{2}{3}$$