Use one answer sheet for each of F1–1, F1–2, F2–1, and F2–2.

Q.1 Find the n-th derivative of the following functions with respect to x, where a is a real number and $a > 0$ and $a \neq 1$.

(1) $\log_e x$

(2) a^x

(3) $x^2 e^x$

(4) $\frac{1}{x^2-1}$

Q.2 Let $z = f(x, y), x = e^u \cos v$, and $y = e^u \sin v$.

Express $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$ in terms of $x, y, \dfrac{\partial^2 z}{\partial x^2}$, and $\dfrac{\partial^2 z}{\partial y^2}$.

Q.3 Solve the following integrals. Derivations must be clearly shown.

(1) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$.

(2) $\displaystyle\int_0^{\infty} \int_0^{\infty} (ax^2 + by^2) e^{-(ax^2+by^2)} dx dy$, where $a > 0$ and $b > 0$. You may use the result of (1).

Q1 (1) $f(x) = \log_e x$

$$f^{(1)}(x) = \frac{1}{x}$$

$$f^{(2)}(x) = \frac{-1}{x^2}$$

$$f^{(3)}(x) = \frac{2}{x^3}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-1)^{n+1} \cdot (n-1)!}{x^n}$$

(2) $f(x) = a^x$

$$f^{(1)}(x) = a^x \log_e a$$

$$f^{(2)}(x) = a^x (\log_e a)^2$$

$$\cdots$$

$$f^{(n)}(x) = a^x (\log_e a)^n$$

(3) $f(x) = x^2 e^x$

$$f^{(1)}(x) = 2xe^x + x^2e^x = (x^2+2x)e^x$$

$$f^{(2)}(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x$$

$$f^{(3)}(x) = (2x+4)e^x + (x^2+4x+2) = (x^2+6x+6)e^x$$

$$f^{(4)}(x) = (2x+6)e^x + (x^2+6x+6) = (x^2+8x+12)e^x$$

$$\cdots$$

$$f^{(n)}(x) = (x^2+2nx+n^2-n)e^x$$

(4) $f(x) = \frac{1}{x^2-1} = \left(\frac{1}{x-1} - \frac{1}{x+1}\right) \cdot \frac{1}{2} = \frac{1}{2}[h(x) - t(x)]$

$h(x) = \frac{1}{x-1}$

$h^{(1)}(x) = \frac{-1}{(x-1)^2}$

$h^{(2)}(x) = \frac{2}{(x-1)^3}$

$h^{(3)}(x) = \frac{-6}{(x-1)^4}$

$\vdots$

$h^{(n)}(x) = \frac{(-1)^n \cdot n!}{(x-1)^{n+1}}$

$t(x) = \frac{1}{x+1}$

$t^{(1)}(x) = \frac{-1}{(x+1)^2}$

$t^{(2)}(x) = \frac{2}{(x+1)^3}$

$t^{(3)}(x) = \frac{-6}{(x+1)^4}$

$\vdots$

$t^{(n)}(x) = \frac{(-1)^n \cdot n!}{(x+1)^{n+1}}$

$\Rightarrow f^{(n)}(x) = \frac{(-1)^n \cdot n!}{2}\left(\frac{1}{(x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right)$

Q.2 $\frac{\partial x}{\partial u} = e^u \cos v = x$, $\frac{\partial x}{\partial v} = -e^u \sin v = -y$

$\frac{\partial y}{\partial u} = e^u \sin v = y$, $\frac{\partial y}{\partial v} = e^u \cos v = x$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial\left(\frac{\partial z}{\partial u}\right)}{\partial y}\cdot\frac{\partial y}{\partial u} = \left(\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2}\right)\cdot x + \left(\frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2}\right)\cdot y$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial\left(\frac{\partial z}{\partial v}\right)}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial\left(\frac{\partial z}{\partial v}\right)}{\partial y}\cdot\frac{\partial y}{\partial v} = \left(-y\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y}\right)(-y) + \left(-\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial y^2}\right)\cdot x$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (x^2+y^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

Q3

(1) Let $I_x = \int_{-\infty}^{\infty} e^{-x^2}dx$, $I_y = \int_{-\infty}^{\infty} e^{-y^2}dy$

$$\Rightarrow (I_x)^2 = I_x \cdot I_y = \iint_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \xlongequal[y=r\sin\theta]{x=r\cos\theta}$$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\cdot r dr d\theta = \frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\infty} e^{-r^2}dr^2$$

$$= \pi(-e^{-r^2})\Big|_0^{\infty} = \pi$$

$$\Rightarrow I_x = \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$$

(2) Let $\begin{cases} x = \dfrac{r\cos\theta}{\sqrt{a}} & r\in(0,+\infty) \\ y = \dfrac{r\sin\theta}{\sqrt{b}} & \theta\in(0,\frac{\pi}{2}) \end{cases}$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\cos\theta}{\sqrt{a}} & \frac{-r\sin\theta}{\sqrt{a}} \\ \frac{\sin\theta}{\sqrt{b}} & \frac{r\cos\theta}{\sqrt{b}} \end{vmatrix} = \frac{r}{\sqrt{ab}}$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^2 e^{-r^2}\cdot\frac{r}{\sqrt{ab}}drd\theta = \frac{1}{\sqrt{ab}}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\infty} r^3 e^{-r^2}dr$$

$$= \frac{\pi}{2\sqrt{ab}}\cdot\frac{1}{4}\int_0^{\infty} e^{-r^2}dr^4 = \frac{\pi}{4\sqrt{ab}}$$