第２問（数学）

次の行列 A，実数ベクトル$\boldsymbol{x}$，関数$f(\boldsymbol{x})$について以下の問いに答えよ．

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad f(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$$

ただし$\boldsymbol{x}^{\mathrm{T}}$は$\boldsymbol{x}$の転置である．

（問 1）次式をみたす正規直交行列Pと対角行列Λを求めよ．

$$P^{\mathrm{T}}AP = \Lambda, \qquad P^{\mathrm{T}}P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

ただし，$\lambda_1 \geq \lambda_2 \geq \lambda_3$とし，行列$P$の第一行目の成分は非負とする．

（問 2）$\|\boldsymbol{x}\|^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1$ のもとで$f(\boldsymbol{x})$の最小値を求めたい．

$$g(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - 1, \qquad L(\boldsymbol{x}, \mu) = f(\boldsymbol{x}) + \mu g(\boldsymbol{x})$$

とするとき，極小点$\boldsymbol{x}$では次式が成り立つことが必要であることが知られている．

$$\frac{\partial L(\boldsymbol{x}, \mu)}{\partial \boldsymbol{x}} = \boldsymbol{0}, \qquad \frac{\partial L(\boldsymbol{x}, \mu)}{\partial \mu} = g(\boldsymbol{x}) = 0$$

(a) $\frac{\partial L(\boldsymbol{x}, \mu)}{\partial \boldsymbol{x}}$を計算せよ．

(b) $f(\boldsymbol{x})$の最小値と，その最小値を与える$\boldsymbol{x}$を求めよ．

（問 3）次の制約式のもとで$f(\boldsymbol{x})$の最小値を求めたい．

$$g_1(\boldsymbol{x}) = (3 \quad -\sqrt{2} \quad -1)\boldsymbol{x} = 0, \; g_2(\boldsymbol{x}) = (3 \quad \sqrt{2} \quad -1)\boldsymbol{x} - 2 = 0$$

(a) $\boldsymbol{x} = P\boldsymbol{y}$の変換を用いて，関数$f(\boldsymbol{x})$と制約式$g_1(\boldsymbol{x}) = 0, g_2(\boldsymbol{x}) = 0$を$y_1, y_2, y_3$により書き換えよ．ただし$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$とする．

(b) $f(\boldsymbol{y})$の最小値と，その最小値を与える$\boldsymbol{y}$を求めよ．

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(1) Let $Ax = \lambda x$

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2+\sqrt{2}-\lambda)(2-\sqrt{2}-\lambda)$$

$\therefore \lambda_1 = 2+\sqrt{2}, \quad \lambda_2 = 2, \quad \lambda_3 = 2-\sqrt{2}$

(i) when $\lambda_1 = 2+\sqrt{2}$

$$|A-\lambda_1 I| = \begin{vmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{vmatrix} \Rightarrow x_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

(ii) when $\lambda_2 = 2$

$$|A-\lambda_2 I| = \begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} \Rightarrow x_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(iii) when $\lambda_3 = 2-\sqrt{2}$

$$|A-\lambda_3 I| = \begin{vmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{vmatrix} \Rightarrow x_3 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2+\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}$$

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(2)(a) $L(x,u) = x^T A x + u(x^T x - 1)$

$$= 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + 2x_3^2 + u(x_1^2 + x_2^2 + x_3^2 - 1)$$

$$= (2+u)x_1^2 + (2+u)x_2^2 + (2+u)x_3^2 - 2x_1x_2 - 2x_2x_3 - u$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2(2+u)x_1 - 2x_2 \\ \dfrac{\partial L}{\partial x_2} = 2(2+u)x_2 - 2x_1 - 2x_3 \\ \dfrac{\partial L}{\partial x_3} = 2(2+u)x_3 - 2x_2 \end{cases}$$

$$\therefore \frac{\partial L(x,u)}{\partial x} = \begin{bmatrix} 4+2u & -2 & 0 \\ -2 & 4+2u & -2 \\ 0 & -2 & 4+2u \end{bmatrix} x$$

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(2)(b) $\begin{cases} \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial L}{\partial x_2} = 0 \\ \frac{\partial L}{\partial x_3} = 0 \\ \frac{\partial L}{\partial u} = x_1^2 + x_2^2 + x_3^2 - 1 = 0 \end{cases}$

$\Rightarrow$ te stationary points are:

$S_1\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$, $S_2\left(-\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}\right)$, $u_1 = -2+\sqrt{2}$

$S_3\left(-\frac{1}{2}, \frac{\sqrt{2}}{2} -\frac{1}{2}\right)$, $S_4\left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$, $u_2 = -2-\sqrt{2}$

$L(S_1, u_1) = 2-\sqrt{2}$, $L(S_2, u_1) = 2-\sqrt{2}$

$L(S_3, u_2) = 2+\sqrt{2}$, $L(S_4, u_2) = 2+\sqrt{2}$

$\Rightarrow f(x)_{min} = 2-\sqrt{2}$

$$x = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} \end{bmatrix}$$

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(b)

Set $L(x,\lambda,\mu) = f(x) + \lambda g_1(x) + \mu g_2(x)$

$$= y^T P^T A P y + \lambda(2y_1 + 2\sqrt{2}y_2) + \mu(2\sqrt{2}y_2 + 2y_3 - 2)$$

$$= (2+\sqrt{2})y_1^2 + 2y_2^2 + (2-\sqrt{2})y_3^2 + 2\lambda y_1 + 2\sqrt{2}\lambda y_2$$

$$+ 2\sqrt{2}\mu y_2 + 2\mu y_3 - 2\mu$$

$$\begin{cases} \dfrac{\partial L}{\partial y_1} = 2(2+\sqrt{2})y_1 + 2\lambda = 0 \\ \dfrac{\partial L}{\partial y_2} = 4y_2 + 2\sqrt{2}\lambda + 2\sqrt{2}\mu = 0 \\ \dfrac{\partial L}{\partial y_3} = 2(2-\sqrt{2})y_3 + 2\mu = 0 \\ \dfrac{\partial L}{\partial \lambda} = 2y_1 + 2\sqrt{2}y_2 = 0 \\ \dfrac{\partial L}{\partial \mu} = 2\sqrt{2}y_2 + 2y_3 - 2 = 0 \end{cases}$$

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(3)(a) $g_1(x) = (3 \quad -\sqrt{2} \quad -1)Py = 2y_1 + 2\sqrt{2}y_2 = 0$

$g_2(x) = (3 \quad \sqrt{2} \quad -1)Py - 2 = 2\sqrt{2}y_2 + 2y_3 - 2 = 0$

$$f(x) = X^TAX = y^TP^TAPy = [y_1 \; y_2 \; y_3]\begin{bmatrix} 2+\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2-\sqrt{2} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= (2+\sqrt{2})y_1^2 + 2y_2^2 + (2-\sqrt{2})y_3^2$$

$\Rightarrow g_1(x) = y_1 + \sqrt{2}y_2 = 0$

$g_2(x) = \sqrt{2}y_2 + y_3 - 1 = 0$

$f(x) = (2+\sqrt{2})y_1^2 + 2y_2^2 + (2-\sqrt{2})y_3^2$

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(3)(b) Set $L(y, \lambda, u) = f(y) + \lambda g_1(x) + u g_2(x)$

$$= y^T A y + \lambda(y_1 + \sqrt{2} y_2) + u(\sqrt{2} y_2 + y_3 - 1)$$

$$= 2y_1^2 - y_1 y_2 - y_1 y_2 + 2y_2^2 - y_2 y_3 - y_2 y_3 + 2y_3^2 + \lambda(y_1 + \sqrt{2} y_2) + u(\sqrt{2} y_2 + y_3 - 1)$$

$$= 2y_1^2 - 2y_1 y_2 + 2y_2^2 - 2y_2 y_3 + 2y_3^2 + \lambda(y_1 + \sqrt{2} y_2) + u(\sqrt{2} y_2 + y_3 - 1)$$

$$\begin{cases} \frac{\partial L}{\partial y_1} = 4y_1 - 2y_2 + \lambda = 0 \\ \frac{\partial L}{\partial y_2} = -2y_1 + 4y_2 - 2y_3 + \sqrt{2}\lambda + \sqrt{2} u = 0 \\ \frac{\partial L}{\partial y_3} = -2y_2 + 4y_3 + u = 0 \\ \frac{\partial L}{\partial \lambda} = y_1 + \sqrt{2} y_2 = 0 \\ \frac{\partial L}{\partial u} = \sqrt{2} y_2 + y_3 - 1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = \frac{-\sqrt{2} - 4}{10 + 2\sqrt{2}} \\ y_2 = \frac{1 + 2\sqrt{2}}{10 + 2\sqrt{2}} \\ y_3 = \frac{6 + \sqrt{2}}{10 + 2\sqrt{2}} \end{cases}$$

$$f(y)_{min} = \frac{20\sqrt{2} + 65}{54 + 20\sqrt{2}}$$

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