数 学 (Mathematics)

6 分野のうちから 3 分野を選び解答すること．選んだ分野毎に解答用紙を別にすること．
Select 3 fields out of the 6 fields and answer the questions. Use a separate answer sheet for each field.

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5. 【確率・統計 (Probability and statistics) 分野】

$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1\}$ とする．連続確率変数の対 (X, Y) の同時密度関数は

$$f(x, y) = \frac{1}{C}(e^{-x} + e^{-y}) \quad (x, y) \in \Omega$$

で与えられるものとする．ただし $C > 0$ は正規化定数である．以下の各問に答えよ．

(1) C の値を求めよ．

(2) X と Y は独立か否か，理由と共に答えよ．

(3) $Y = 0$ の条件の下での X の期待値を求めよ．

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Let (X, Y) be a pair of continuous random variables given by the joint probability density function

$$f(x, y) = \frac{1}{C}(e^{-x} + e^{-y}) \quad (x, y) \in \Omega$$

where $C > 0$ denotes the normalizing constant. Answer the following questions.

(1) Compute the value of C.

(2) Establish whether or not X and Y are independent.

(3) Find the expectation of X under the condition of $Y = 0$.

(1) $\int_0^1\int_0^1 f(x,y)\,dx\,dy = \frac{1}{c}\int_0^1\int_0^1 (e^{-x}+e^{-y})\,dx\,dy$

$= \frac{1}{c}\left(\frac{-2}{e}+2\right) = 1$

$\Rightarrow \quad c = 2-\frac{2}{e}$

(2) $f_X(x) = \int f(x,y)\,dy = \frac{1}{c}\int (e^{-x}+e^{-y})\,dy$

$= \frac{1}{c}(ye^{-x}-e^{-y})$

$f_Y(y) = \int f(x,y)\,dx = \frac{1}{c}\int (e^{-x}+e^{-y})\,dx$

$= \frac{1}{c}(-e^{-x}+xe^{-y})$

$f_X(x)f_Y(y) = \frac{1}{c^2}\left(-ye^{-2x}+xye^{-(x+y)}+e^{-(x+y)}-xe^{-2y}\right)$

$\neq f(x,y)$

thus X and Y are not independent.

(3) $P(X \le x \mid Y=0) = \frac{1}{c}\int_0^x (e^{-x}+1)\,dx$

$$= \frac{1}{c}(1+x-e^{-x})$$

$$E[X \le x \mid Y=0] = \int_{-\infty}^{\infty} x\,P(X \le x \mid Y=0)$$

$$= \frac{1}{c}\int_0^1 (x+x^2-xe^{-x})\,dx$$

$$= \frac{1}{c}\left[\int_0^1 (x+x^2)\,dx - \int_0^1 xe^{-x}\,dx\right]$$

$$= \frac{1}{c}\left[\frac{5}{6} - \left(-xe^{-x}-e^{-x}\right)\Big|_0^1\right]$$

$$= \frac{1}{2-\frac{2}{e}} \times \left(-\frac{1}{6}+2e^{-1}\right)$$

$$= \frac{12-e}{12e-12}$$